Complexified Ward Identity in pure Yang-Mills theory at tree-level

Gang Chen,[1]

[1] *Department of Physics, Nanjing University*
22 Hankou Road, Nanjing 210098, China

(Dated: March 2, 2013)

Abstract

Up until now, the BCFW technique has been a widely used method in getting the amplitudes in various theories. Usually, the vanishing of the boundary term is necessary for the efficiency of the method. However, there are also many kinds of amplitudes which will have boundary terms. Hence it will be nice if it is possible to get the boundary terms in an efficient manner. As is well-known, in gauge theory the Ward identity imposes constraints on the form of the amplitude. In [1], we studied the Ward identity with a pair of shifted lines and the implied recursion relations. In this article, we discuss the complexified Ward identity in more detail. In particular we give a proof of the complexified Ward identity directly from the Feynman rules in Feynman-Lorentz gauge. Furthermore, we give more examples in calculating the one-line off-shell currents using this technique.

PACS numbers: 11.15.Bt, 12.38.Bx, 11.25.Tq

I. INTRODUCTION

In the early development of Quantum mechanics, A. Einstein had proposed the possibility of constructing the Quantum theory from a purely algebraic theory. At that time, nobody knows how to obtain the basis of such a theory. In Quantum Field Theory (QFT), it seems even more difficult for this aim except for some special case such as Topological field theory. Recent years, there are exiting progress (BCFW) on the amplitudes in QFT, which is helpful for constructing an QFT from algebra system. At tree level, the amplitudes in pure Yang-Mills theory can be taken as rational functions of external momenta and external states in spinor forms [2–7]. Such rational functions are probable to study in detail in algebra system now. According to this, BCFW recursion relation was proposed and developed in [8–10]. It was then proven in [11] using the singularity properties of the tree-level on-shell amplitudes. For the theory with massive fields [12–16], the amplitudes are also rational functions of external momenta and states in spinor forms.

At loop level, although the total amplitudes are not rational functions in general any more, they can be decomposed in some basic scalar integrals with coefficients being rational functions of external spinors [17, 18]. The coefficients structures are also studied deeply in [19–21]. At loop level, the integrands of the amplitudes are rational functions of the external spinors and integral momenta. For the N=4 planar super Yang-Mills theory, [22] give an explicit recursive formula for the all-loop integrand for scattering amplitudes.

In gauge theory, Ward identity is a constraint for the amplitudes at any loop-level. Hence it lead a algebra constraint for the rational parts in the amplitudes. In this article, we discuss more about Ward identity. We complexify the momenta of a pair of external lines in a proper way and give a proof of the Ward identity with complex external momenta using Feynman rules. In BCFW formalism, the vanishing of the boundary term is necessary for the application of the recursion relation. However, there are still various kinds of amplitudes which do have boundary terms. In this article, we will use the complexified Ward identity to determine the amplitudes in gauge theory. For the poles at finite position, the residues are the same as those in BCFW recursion relations. Moreover, this Ward identity will lead to new forms for the boundary terms, which can be obtained by another recursion relation. We will focus on the vector off-shell current which is the amplitude with one external line amputated and its momenta extended to off-shell. The method can also be extended to the tensor off-shell currents with several external off-shell lines. Our method is particularly useful for the case that the on-shell lines do not have the same helicity structures. In this sense, our technique is complementary to

the off-shell current recursion relation presented in [4, 5].

II. COMPLEXIFIED WARD IDENTITY

In an interacting gauge theory, the Ward identity is a consequence of the gauge current conservation . As a result, the tensor currents vanish when contracted with all the momenta of the external off-shell lines

$$k^1_{\mu_1} k^2_{\mu_2} \cdots k^m_{\mu_m} \mathcal{A}^{\mu_1\mu_2\cdots\mu_m} = 0, \tag{1}$$

where m is the number of the external off-shell lines and $k^i_{\mu_i}$ is the momenta of the external off-shell lines.. The tensor current $\mathcal{A}^{\mu_1\mu_2\cdots\mu_m}$ is defined to be the amplitude with removing the propagators of the external off shell lines, while in [6], they keep the propagators of the external off-shell lines in the definition. This subtle difference will lead the current formalisms in this article seem to be different with those in [23]. However actually our results are identical with those in [23].

As long as the current conservation is not broken by quantum corrections, the identity holds at each level of the perturbative expansion. Furthermore, at tree level, the Ward identity holds even when the momenta are complexified with the on-shell condition untouched. Hence it contains much more information about the off-shell currents than just the current conservation. There is a simple proof for the complexified Ward identity directly according to the Feynman rules in Lorentz-Feynman gauge, where the tree level 2, 3 and 4 point vertices take the forms

$$\begin{aligned}
V^2_{\mu\nu} &= -i\frac{\eta_{\mu\nu}}{k^2} \\
V^3_{\mu_1\mu_2\mu_3} &= \frac{i}{\sqrt{2}}\left(\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\right) \\
V^4_{\mu_1\mu_2\mu_3\mu_4} &= \frac{i}{2}\left(2\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}\right).
\end{aligned} \tag{2}$$

We prove the complexified Ward identity by induction. Firstly, it is easy to check directly that the complexified Ward identity still holds for the currents with two and three on-shell lines. Secondly, we assume the complexified Ward identity holds for the currents with the external on-shell lines less than N. Then we need to prove the currents with N on-shell lines are conserved.

To this end, we first the classify the diagrams according to the types of the vertex $\bar{V}$ connecting directly to the external off-shell line and of the nearest two vertices V_a and

V_b of on the left and right hand the off-shell line. Hence in total, there are five kinds of diagrams.

Case 1: $\bar{V}$ is 4-gluon vertex. Then we denote the result of the current contracted with the momentum of external off-shell line as $F^N(n_1, n_2, n_3)$, where n_1, n_2, n_3 are the positive numbers of external lines in J_1, J_2, J_3 respectively. And $n_1 + n_2 + n_3 = N$. Similarly for the other four cases;

Case 2: $\bar{V}$, V_a and V_b are 3-gluon vertices. Then we denote the result as $T^N(n_1, n_2 \vdots m_1, m_2)$. The v-dots denotes the position of the off-shell external line;

Case 3: $\bar{V}$, V_a are 3-gluon vertices and and V_b is 4-gluon vertex. Then we denote the results as $T^N(n_1, n_2 \vdots m_1, m_2, m_3)$;

Case 4: $\bar{V}$, V_b are 3-gluon vertices and and V_a is 4-gluon vertex. Then we denote the results as $T^N(n_1, n_2, n_3 \vdots m_1, m_2)$;

Case 5: $\bar{V}$ is 3-gluon vertex and and V_a, V_b are 4-gluon vertices. Then we denote the results as $T^N(n_1, n_2, n_3 \vdots m_1, m_2, m_3)$.

Then according to inductive assumption about the current conservation for fewer external on-shell lines, when multiplied with the external off-shell momentum, in case 2-5, $\bar{V}^3_{\mu_1\mu_2\mu_3} k_3^{\mu_3} = \frac{i}{\sqrt{2}} \eta_{\mu_1\mu_2}(k_2^2 - k_1^2)$. We divide it into $\bar{V}^3_{\mu_1\mu_2\mu_3} k_3^{\mu_3} = V^R_{\mu_1\mu_2} k_2^2 - V^L_{\mu_1\mu_2} k_1^2 \equiv V^R_{\mu_1\mu_2} + V^{-L}_{\mu_1\mu_2}$. Correspondingly, $T^N = T^N_R + T^N_{-L}$.

To make the cancelation obvious, we arrange the terms for all the diagrams as follows

$$
\begin{array}{ll}
 & T^N(1\vdots 1, N-2)_R + T^N(1\vdots 2, N-3)_R + \cdots + T^N(1\vdots N-2, 1)_R \\
+ & T^N(1\vdots 1, 1, N-3)_R + T^N(1\vdots 1, 2, N-4)_R + \cdots + T^N(1\vdots N-3, 1, 1)_R \\
\hline
+ & T^N(1, 1\vdots 1, N-3)_{R-L} + T^N(1, 1\vdots 2, N-4)_{R-L} + \cdots + T^N(1, 1\vdots N-3, 1)_{R-L} \\
+ & T^N(1, 1\vdots 1, 1, N-4)_{R-L} + T^N(1, 1\vdots 1, 2, N-5)_{R-L} + \cdots + T^N(1, 1\vdots N-4, 1, 1)_{R-L} \\
\hline
+ & T^N(1, 2\vdots 1, N-4)_{R-L} + T^N(1, 2\vdots 2, N-5)_{R-L} + \cdots + T^N(1, 2\vdots N-4, 1)_{R-L} \\
+ & T^N(1, 2\vdots 1, 1, N-5)_{R-L} + T^N(1, 2\vdots 1, 2, N-6)_{R-L} + \cdots + T^N(1, 2\vdots N-5, 1, 1)_{R-L} \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
+ & T^N(1, 1, 1\vdots 1, N-4)_{R-L} + T^N(1, 1, 1\vdots 2, N-5)_{R-L} + \cdots + T^N(1, 1, 1\vdots N-4, 1)_{R-L} \\
+ & T^N(1, 1, 1\vdots 1, 1, N-5)_{R-L} + T^N(1, 1, 1\vdots 1, 2, N-6)_{R-L} + \cdots + T^N(1, 1, 1\vdots N-5, 1, 1)_{R-L} \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
+ & T^N(2, 1\vdots 1, N-4)_{R-L} + T^N(2, 1\vdots 2, N-5)_{R-L} + \cdots + T^N(2, 1\vdots N-4, 1)_{R-L} \\
+ & T^N(2, 1\vdots 1, 1, N-5)_{R-L} + T^N(2, 1\vdots 1, 2, N-6)_{R-L} + \cdots + T^N(2, 1\vdots N-5, 1, 1)_{R-L} \\
\hline
\vdots & \\
\vdots & \\
\hline
+ & T^N(N-2, 1\vdots 1)_{-L} \\
\vdots & \\
+ & T^N(1, N-2\vdots 1)_{-L} \\
+ & T^N(1, 1, N-3\vdots 1)_{-L} \\
+ & T^N(1, 2, N-4\vdots 1)_{-L} \\
\vdots & \\
+ & T^N(N-3, 1, 1\vdots 2, 1)_{-L} \\
\hline
+ & \displaystyle\sum_{n_1, n_2, n_3} F^N(n_1, n_2, n_3)
\end{array}
\qquad (3)
$$

Under the recursive assumption, we can verify directly that the summations of diagrams as shown in Fig.1 and Fig.2 will vanish. That is

$$
\begin{aligned}
& T(\{n_1\}\vdots n_2, n_3)_R + T(n_1, n_2\vdots\{n_3\})_{-L} + F(n_1, n_2, n_3) = 0 \\
& T(\{n_1\}\vdots n_2, n_3, n_4)_R + T(n_1, n_2, n_3\vdots\{n_4\})_{-L} = 0,
\end{aligned}
\qquad (4)
$$

where $\{n\}$ denotes all the possible divisions of n into (i, j) and (i, j, k). Then we can find, in (3), in each box between two horizontal lines, the summation of the L-terms with fixed

pairs (n_1, n_2) and triples (m_1, m_2, m_3) to the left of the off-shell line are $T(n_1, n_2 \vdots \{n_3\})_L$ and $T(m_1, m_2, m_3 \vdots \{m_4\})_L$ respectively. And similarly the summation of the R-terms with fixed pairs (i_1, i_2) and triples (j_1, j_2, j_3) to the right of the off-shell line are $T(\{i_3\} \vdots i_1, i_2)_R$ and $T(\{j_4\} \vdots j_1, j_2, j_3)_R$. Then we can arrange all the terms into the groups as in (4). Hence they cancel exactly with each other.

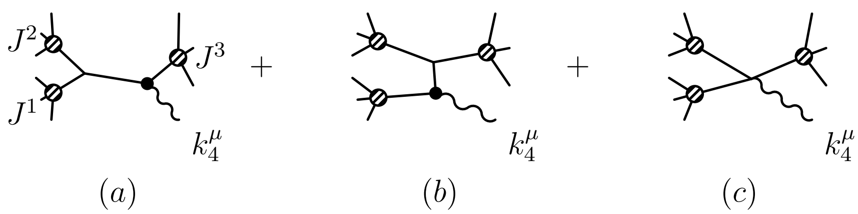

FIG. 1: In thees diagrams we use $\bullet$ to denote the $\bar{V}_{-L}$ and $\bar{V}_R$ to (a) and (b) respectively.

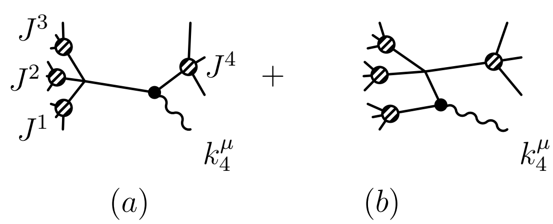

FIG. 2: Similarly as Fig.1, the $\bullet$ denote the $\bar{V}_{-L}$ and $\bar{V}_R$ for (a) and (b) respectively.

Now we shift the momenta of a pair of lines i, v, where i is one on-shell line with external state ϵ_i and v is the off-shell line. The momentum shift is chosen such as to keep the momentum conserved and the i-line on-shell. For example, we can shift the momenta as $\hat{p}_i = p_i - z\eta_i$ and $\hat{p}_v = p_v - z\eta_i$, where z is an arbitrary complex parameter and η_i is a four-vector which satisfies $\eta_i \cdot \epsilon_i = 0$. Then we get a complexified form of the Ward identity. Acting with the first order derivative with respect to z on the complexified Ward identity, we obtain

$$\hat{\mathcal{A}}(z)_\mu \eta_i^\mu = -\hat{p}_v^\mu \frac{d\hat{\mathcal{A}}(z)_\mu}{dz}, \tag{5}$$

where $\hat{\mathcal{A}}(z)_\mu$ denotes the complexified vector off-shell current.

If the shifted on-shell line is in helicity state $\epsilon_i^+ = \frac{\mu \tilde{\lambda}_i}{\langle \mu, \lambda_i \rangle}$, we shift the momenta as $\lambda_i \to \lambda_i - z\lambda_v$, $\tilde{\lambda}_v \to \tilde{\lambda}_v + z\tilde{\lambda}_i$, and $\eta_i = \lambda_v \tilde{\lambda}_i$. It is easy to see that such momenta shift only lead to a z-dependent in the denominator of ϵ_i^+. To avoid the unphysical pole in the

denominator of ϵ_i^+, we choose the reference spinor as $\mu = \lambda_v$. Then ϵ_i^+ do not dependent on z even under the momenta shift and we do not get a extra contribution from the external wave function ϵ_i^+ in the left hand of (5). Hence the choice of the momenta shift and the reference spinor above are convenient for the calculation in practice. Similarly, for the negative state $\epsilon_i^- = \frac{\lambda_i \tilde{\mu}}{\langle \tilde{\lambda}_i, \tilde{\mu} \rangle}$, we can shift its momentum as $\tilde{\lambda}_i \to \tilde{\lambda}_i - z\tilde{\lambda}_v$, $\lambda_v \to \lambda_v + z\lambda_i$. For the same reason, the reference spinor is taken as $\tilde{\mu} = \tilde{\lambda}_v$.

For pure Yang-Mills theory, we can expand the $\hat{\mathcal{A}}(z)_\mu$ with respect to z as

$$\hat{\mathcal{A}}(z)_\mu = A^1_\mu z + A^0_\mu + A^{-1_a}_\mu \frac{1}{z-a} + A^{-1_b}_\mu \frac{1}{z-b} + \cdots \tag{6}$$

According to (5), the term A^0_μ should not contribute to the amplitude. Comparing (5) with (6), it is clean that the boundary term should be

$$A^0 \cdot \eta_i = -A^1 \cdot p_v. \tag{7}$$

Therefore A^0, which is hard to obtain directly, can be transformed to the A^1 which can be obtained by a new recursion relation.

III. OFF-SHELL VECTOR CURRENT FROM THE WARD IDENTITY

Now we apply our technology to the off-shell vector currents. Without loss of generality, we choose the shifted on-shell line to be of + helicity. Under the gauge described above, the overall behavior of the currents is z^1 when $z \to \infty$. According to eq. (5), we get

$$\begin{aligned} \eta_i^\mu \hat{A}_\mu &= -\hat{p}_v^\mu A^1_\mu + \sum_m \sum_h \hat{p}_v^\mu \frac{A_L^h(z_m)(A_R^{\tilde{h}})_\mu(z_m)}{2P_m \cdot \eta_i (z-z_m)^2} \\ &= -\hat{p}_v^\mu A^1_\mu + \sum_m \sum_h \frac{A_L^h(z_m)(A_R^{\tilde{h}})_\mu(z_m)\eta_i^\mu}{2P_m \cdot \eta_i (z-z_m)}, \end{aligned} \tag{8}$$

where the we choose the off-shell line in $A_R^{\tilde{h}}$. Since the factor $A_L^{k_m}(z_m)$ vanishes, we only need to take the summation over $(h, \tilde{h}) \in \{(+,-),(-,+),(r,k)\}$[23]. Then the current's projection on η_i can be obtained by setting $z = 0$ in (8)

$$\eta_i^\mu A_\mu = -p_v^\mu A^1_\mu + \sum_m \sum_h \frac{A_L^h(z_m)(A_R^{\tilde{h}})_\mu(z_m)\eta_i^\mu}{2P_m \cdot \eta_i (-z_m)}. \tag{9}$$

Using the complexified Ward Identity, we can hence transform the unknown term A^0 into A^1 which can also be obtained by a new recursion relation. In fact when we choose

the momentum shift and the gauge of the external on-shell states as discussed above, the wave function does not depend on z. Moreover, the four point vertices do not contain the momentum factor, they are also independent of z. The z dependent terms only come from the three point vertices and propagators in the complex lines from external line i to v. $A^1_\mu = (\frac{d\hat{A}(z)_\mu}{dz})^0$ then gets two kinds of contributions as shown in Fig. 3. When acting

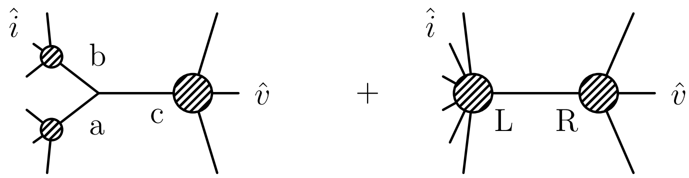

FIG. 3: The Feynman diagrams which will contribute to the boundary terms

with $\frac{d}{dz}$ on the propagators and extracting the zeroth order terms in z, we get

$$i(A_L^{\nu_L})^1 \frac{g_{\nu_L \nu_R}}{2P_m \cdot \eta_i} (A_R^{\nu_R \mu})^1. \tag{10}$$

When acting with $\frac{d}{dz}$ on the three point vertices, we get

$$\frac{1}{\sqrt{2}} \frac{A_a \cdot (A_b)^1 \; \eta_i \cdot (A_c^\mu)^1}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2} - \sqrt{2} \frac{(A_b)^1 \cdot (A_c^\mu)^1 \; \eta_i \cdot A_a}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2}. \tag{11}$$

Finally, we take the summation of all the complexified three-point vertices and the propagators

$$(A^\mu)^1 = \sum_{a,b,c} \left(\frac{1}{\sqrt{2}} \frac{A_a \cdot (A_b)^1 \; \eta_i \cdot (A_c^\mu)^1}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2} - \sqrt{2} \frac{(A_b)^1 \cdot (A_c^\mu)^1 \; \eta_i \cdot A_a}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2} \right) + \sum_m i \frac{(A_L)^1 \cdot (A_R^\mu)^1}{2p_m \cdot \eta_i}. \tag{12}$$

To complete the recursion relation, we need to know the coefficients of order z in the tensor off-shell currents $(\hat{A}^{\nu\mu})^1$. It is similar to the $(A^\mu)^1$,

$$(A^{\nu\mu})^1 = \frac{1}{\sqrt{2}} \frac{A_a \cdot (A_b^\nu)^1 \; \eta_i \cdot (A_c^\mu)^1}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2} - \sqrt{2} \frac{(A_b^\nu)^1 \cdot (A_c^\mu)^1 \; \eta_i \cdot A_a}{2p_b \cdot \eta_i \; 2p_c \cdot \eta_i \; p_a^2} + i \frac{(A_L^\nu)^1 \cdot (A_R^\mu)^1}{2p_m \cdot \eta_i}. \tag{13}$$

In (9), there are new non-vanishing objects which can be taken as the off-shell amplitudes with one external states of the on-shell lines replaced by its momentum. On proceeding several recursive steps, we get a general form $\mathcal{A}^\mu(\cdots, k_{i_1}, \cdots, k_{i_j}, \cdots, k_{i_N})$, where we omit on-shell states with N denotes the total number of the replaced on-shell lines.

Inevitably, we will need to shift the momentum of such line together with the off-shell line. The boundary term then can not be obtained as discussed above. Under the

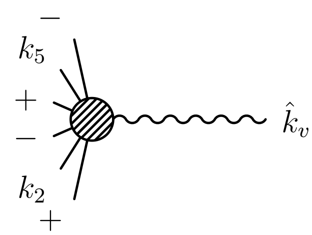

FIG. 4: Color-ordered Feynman rules.

momentum shifting, $\lambda_i \to \lambda_i - z\lambda_v, \tilde{\lambda}_v \to \tilde{\lambda}_v + z\tilde{\lambda}_i$, the vector currents are of the form $\hat{k}_i^\nu \hat{A}_{\nu\mu}$. The shifted momentum contains a factor proportional to z. This will lead to $(\hat{A}^\mu)^0$ contributing to the boundary term. Luckily we only need to know the amplitude when $z \to 0$. As it is obvious that $\hat{k}_i^\nu \hat{A}_{\nu\mu}|_{z=0} = k_i^\nu \hat{A}_{\nu\mu}|_{z=0}$, we only need to consider $k_i^\nu \hat{A}_{\nu\mu}|_{z=0}$.

There is a similar identity for calculating this kind of currents

$$\frac{k_i^\nu \hat{A}_{\nu\mu} \hat{K}^\mu}{[\tilde{\lambda}_m, \tilde{\lambda}_v]} = 0, \tag{14}$$

which can be deduced from the Ward identity,

$$\begin{aligned} \frac{\hat{k}_i^\nu \hat{A}_{\nu\mu} \hat{K}^\mu}{[\tilde{\lambda}_m, \tilde{\lambda}_v]} &= 0, \\ \frac{\eta_i^\nu \hat{A}_{\nu\mu} \hat{K}^\mu}{[\tilde{\lambda}_m, \tilde{\lambda}_v]} &= \epsilon_i^\nu \hat{A}_{\nu\mu} \hat{K}^\mu = 0. \end{aligned} \tag{15}$$

From (14), we get a similar recursion relation for $k_i^\nu A_{\nu\mu} \eta_i^\mu$. Similarly, for another momentum shift $\tilde{\lambda}_i \to \tilde{\lambda}_i - z\tilde{\lambda}_v, \lambda_v \to \lambda_v + z\lambda_i$, we get a recursion relation for $k_i^\nu A_{\nu\mu} \tilde{\eta}_i^\mu$. Now we have successfully expressed the boundary term in the amplitude as terms composed of the off-shell amplitudes with fewer external lines. The component of the off-shell vector current in the momentum direction vanishes according to the Ward identity. To get the full off-shell vector current, we hence need to project onto three linearly independent directions, non of which is parallel with the momentum. We can obtain all of them using the same procedure above.

Without loss of generality, we choose the helicities of lines (i, j, k) to be $(+, +, -)$ respectively. In the same way, we can obtain $\eta_j \cdot\ A$ and $\tilde{\eta}_k \cdot A$ where $\eta_j = \lambda_v \tilde{\lambda}_j$ and $\tilde{\eta}_k = \lambda_k \tilde{\lambda}_v$. It is convenient to write the vector current as $A^\mu = x_1 \eta_i^\mu + x_2 \eta_j^\mu + x_3 \tilde{\eta}_k^\mu + x_4 K_v^\mu$.

We can then determine the off-shell vector currents by solving the following four equations

$$\begin{aligned} e_{ik}x_3 + e_{iv}x_4 &= \eta_i \cdot A \\ e_{jk}x_3 + e_{jv}x_4 &= \eta_j \cdot A \\ e_{ki}x_1 + e_{kj}x_2 + e_{kv}x_4 &= \eta_k \cdot A \\ e_{vi}x_1 + e_{vj}x_2 + e_{vk}x_3 + e_{vv}x_4 &= 0, \end{aligned} \tag{16}$$

where e_{ij} are the inner products of the basis vectors and the lower indices denote the corresponding basis. Hence to get the full vector current A^μ, we only need to choose three different kinds of momentum shift, by choosing various external lines or shifts, such that the shifted momenta are linearly independent. This is possible for any vector currents with three external on-shell lines.

IV. EXAMPLES

Before proceeding on the concrete examples, we summarize the procedure of calculating the vector currents. When the helicities of on-shell lines in the currents are same, it is direct to write the expressions of the currents as in [4, 6]. For the case when the external on-shell lines are of mixed helicity structures, we choose the reference spinors to be λ_v and $\tilde{\lambda}_v$ for the $+$ and $-$ states respectively. We then choose three kinds of momentum shift such that all the external states are independent on the shifting parameter z and the shifting momenta are linear independent with each other and not parallel with the momentum of the external off-shell line. Under each kind of momenta shift, we then get one component of the current. Combining with the Ward identity, we can recover the whole vector current. Each non-vanishing component of the complexified current is composed of a boundary term and single pole terms. For single pole terms, we get them by the BCFW recursion relation. The referent vector, which will appear in cutting the internal propagators, are chosen to be $\lambda_v\tilde{\lambda}_v$ for the simplification of calculations. For the boundary term, we get it by the new recursion relation in Section III.

A. currents with two on-shell lines

There are two building blocks for the recursion relations. One is the three-point on-shell amplitude. The form can be find in earlier papers [4, 12]. The other is the off-shell current with three external lines with one off-shell external line and two on-shell external lines. The external vector of every on-shell line can be physical state or its momentum.

The off-shell currents take the following forms,

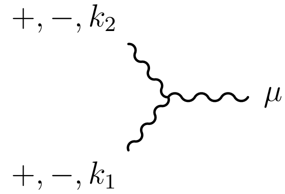

FIG. 5: Color-ordered Feynman rules.

$$
\begin{aligned}
A(k_1, k_2, \mu) &= \frac{i}{\sqrt{2}} k_1 \cdot k_2 (k_2 - k_1)_\mu \\
A(\pm, k_2, \mu) &= \frac{i}{\sqrt{2}} \left(k_2 \cdot \epsilon_1^\pm (k_2 + k_1)_\mu - 2k_1 \cdot k_2 (\epsilon_1^\pm)_\mu \right) \\
A(k_1, \pm, \mu) &= \frac{i}{\sqrt{2}} \left(-k_1 \cdot \epsilon_2^\pm (k_2 + k_1)_\mu + 2k_1 \cdot k_2 (\epsilon_2^\pm)_\mu \right)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
A(\pm, \pm, \mu) &= \frac{i}{\sqrt{2}} \left(2k_2 \cdot \epsilon_1^\pm (\epsilon_2^\pm)_\mu - 2k_1 \cdot \epsilon_2^\pm (\epsilon_1^\pm)_\mu \right) \\
A(\mp, \pm, \mu) &= \frac{i}{\sqrt{2}} \left(\epsilon_1^\mp \cdot \epsilon_2^\pm (k_1 - k_2)_\mu + 2k_2 \cdot \epsilon_1^\mp (\epsilon_2^\pm)_\mu - 2k_1 \cdot \epsilon_2^\pm (\epsilon_1^\mp)_\mu \right).
\end{aligned} \tag{18}
$$

There is a direct justification of equation (7) for three-line off-shell amplitudes. We take $A(+, -, \mu)$ as an example. We choose the $+$-momentum shift for lines 1 and 3. The reference spinors of the external lines are the same as above. It is easy to show that $A^0 \cdot \eta_1 = -A^1 \cdot k_3$:

$$
\begin{aligned}
A^0 \cdot \eta_1 &= \frac{i}{\sqrt{2}} \left(-\epsilon_1^+ \cdot \epsilon_2^- k_2 \cdot \eta_1 + 2k_2 \cdot \epsilon_1^+ \epsilon_2^- \cdot \eta_1 \right) = \frac{i}{\sqrt{2}} \epsilon_1^+ \cdot \epsilon_2^- k_2 \cdot \eta_1 \\
-A^1 \cdot k_3 &= \frac{i}{\sqrt{2}} \left(\epsilon_1^+ \cdot \epsilon_2^- k_3 \cdot \eta_1 - 2k_3 \cdot \epsilon_1^+ \epsilon_2^- \cdot \eta_1 \right) = \frac{i}{\sqrt{2}} \epsilon_1^+ \cdot \epsilon_2^- k_2 \cdot \eta_1.
\end{aligned} \tag{19}
$$

B. currents with three on-shell lines

The first non-trivial example is an amplitude with four lines, one of which is off-shell. For concreteness, we take the amplitude as $A(1^+, 2^+, 3^-, \mu_4)$. As stated above, the reference momenta and spinors are taken as $k_r = \lambda_r \tilde{\lambda}_r$ and $\mu_1 = \mu_2 = \lambda_r, \tilde{\mu}_3 = \tilde{\lambda}_r$. The shifted momenta are $\eta_1 = \lambda_r \tilde{\lambda}_1, \eta_2 = \lambda_r \tilde{\lambda}_2, \tilde{\eta}_3 = \tilde{\lambda}_r \lambda_3$ for the states $\epsilon_1^+ = \frac{\lambda_r \tilde{\lambda}_1}{\langle \lambda_r, \lambda_1 \rangle}, \epsilon_2^+ = \frac{\lambda_r \tilde{\lambda}_2}{\langle \lambda_r, \lambda_2 \rangle}, \epsilon_3^- = \frac{\lambda_3 \tilde{\lambda}_r}{[\tilde{\lambda}_3, \tilde{\lambda}_r]}$ respectively. Through these shifts, one can get the components $A \cdot$

$\eta_1, A\cdot\eta_2, A\cdot\tilde{\eta}_3$ of the off-shell amplitude vector. We will compare the results from our methods with those from the usual Feynman rules for $A\cdot\eta_1$. Similar discussions are valid for $A\cdot\eta_2, A\cdot\tilde{\eta}_3$. In Feynman rules the four-line amplitude involves the following three diagrams. Figure (a) contributes a term

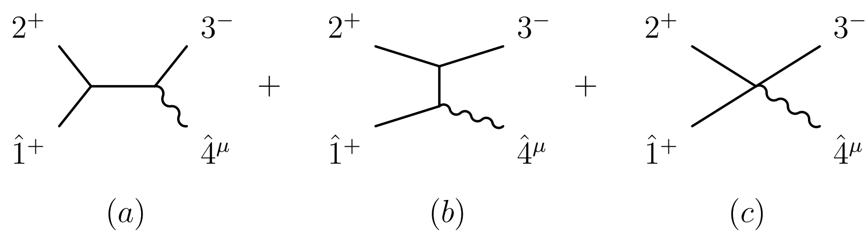

FIG. 6: Color-ordered Feynman rules.

$$\frac{A(1^+,2^+,\nu_1)g^{\nu_1\nu_2}A(\nu_2,3^-,\mu)\eta^\mu}{p_{12}^2}. \tag{20}$$

Figure (b) is

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\frac{\epsilon_1\cdot A(\mu,2^+,3^-)(k_1-k_{14})\cdot\eta_1+2\eta_1\cdot A(\mu,2^+,3^-)k_{14}\cdot\epsilon_1}{p_{23}^2}\\
=&\frac{1}{\sqrt{2}}\frac{\eta_1\cdot A(\mu,2^+,3^-)(k_1+k_{14})\cdot\epsilon_1}{p_{23}^2}\\
=&\frac{-1}{\sqrt{2}}\frac{\eta_1\cdot A(\mu,2^+,3^-)k_4\cdot\epsilon_1}{p_{23}^2}.
\end{aligned} \tag{21}$$

Figure (c) vanishes for $A\cdot\eta_1$. Using our methods, we get the off-shell amplitude as

$$\begin{aligned}
A\cdot\eta_1 &= -p_4\cdot A^1+\frac{A(\hat{1}^+,2^+,\hat{m}^-)A(\hat{m}^+,3^-,\hat{4}^\mu)\eta^\mu}{2P_m\cdot\eta(-z_m)}\\
&= -\frac{1}{\sqrt{2}}\frac{\eta\cdot A(\mu,2^+,3^-)\epsilon_{\hat{1}^+}\cdot p_4}{p_{23}^2}+\frac{A(\hat{1}^+,2^+,\hat{m}^-)A(\hat{m}^+,3^-,\hat{4}^\mu)\eta^\mu}{p_{12}^2}.
\end{aligned} \tag{22}$$

The first term is equal to (21). The second term can be rewritten as

$$\begin{aligned}
&\frac{A^{z_m}(\hat{1}^+,2^+,\hat{m}^-)A^{z_m}(\hat{m}^+,3^-,\hat{4}^\mu)\eta^\mu}{p_{12}^2}=\frac{A^{z_m}(\hat{1}^+,2^+,\nu_1)g^{\nu_1\nu_2}A^{z_m}(\nu_2,3^-,\hat{4}^\mu)\eta^\mu}{p_{12}^2}\\
=&\frac{\left(A(1^+,2^+,\nu_1)+\frac{i}{\sqrt{2}}\left(z_m\epsilon_1^+\cdot\epsilon_2^+(-\eta)_{\nu_1}+2z_m\eta_1\cdot\epsilon_2^+(\epsilon_1^+)_{\nu_1}\right)\right)g^{\nu_1\nu_2}A^{z_m}(\nu_2,3^-,\hat{4}^\mu)\eta^\mu}{p_{12}^2}\\
=&\frac{A(1^+,2^+,\nu_1)g^{\nu_1\nu_2}\left(A(\nu_2,3^-,4^\mu)\eta_1^\mu+\frac{i}{\sqrt{2}}\left(z_m(\epsilon_2^+)_{v_2}(-\eta_1)_\mu\eta_1^\mu+2z_m(\eta_1)_{v2}\eta_1\cdot\epsilon_3^-\right)\right)}{p_{12}^2}\\
=&\frac{A(1^+,2^+,\nu_1)g^{\nu_1\nu_2}A(\nu_2,3^-,\mu)\eta^\mu}{p_{12}^2},
\end{aligned} \tag{23}$$

which is exactly the same as (20). When shifting the momenta of a pair of lines $(2,4)$, we get

$$\eta_2 \cdot A \;=\; -p_4 \cdot A^1 + \frac{A(1^+,\hat{2}^+,\hat{m}_{12}^-)A(\hat{m}_{34}^+,3^-,\hat{4}^\mu)\eta_2^\mu}{2p_{34}\cdot\eta_2(-z_{m_{12}})} + \frac{A(\hat{2}^+,3^-,\hat{m}_{23}^+)A(\hat{m}_{14}^-,\hat{4}^\mu,1^+)\eta_2^\mu}{2p_{14}\cdot\eta_2(-z_{m_{12}})} + \frac{A(\hat{2}^+,3^-,\hat{m}_{23}^v)A(\hat{m}_{14}^k,\hat{4}^\mu,1^+)\cdot\eta_2^\mu}{2p_{14}\cdot\eta_2(-z_{m_{12}})}, \quad (24)$$

where

$$-p_4\cdot A^1 = \frac{1}{\sqrt{2}}\frac{\eta_2^\mu A^1(m_{14}^\mu,\hat{4}^\nu,1^+)p_4^\nu\epsilon_2^+\cdot\epsilon_3^-}{2p_{14}\cdot\eta_2(-z_{m_{12}})} - i\frac{A^1(\hat{2}^+,3^-,m_{23}^\mu)A^1(m_{14}^\mu,\hat{4}^\nu,1^+)p_4^\nu}{2p_{14}\cdot\eta_2} \quad (25)$$

For shifting the momenta the pair of lines $(3,4)$, we get

$$\tilde{\eta}_3\cdot A \;=\; -p_4\cdot A^1 + \frac{A(2^+,\hat{3}^-,\hat{m}_{23}^-)A(\hat{m}_{14}^+,\hat{4}^\mu,1^+)\eta_3^\mu}{2p_{14}\cdot\tilde{\eta}_3(-z_{m_{14}})} + \frac{A(2^+,\hat{3}^-,\hat{m}_{23}^+)A(\hat{m}_{14}^-,\hat{4}^\mu,1^+)\eta_3^\mu}{2p_{14}\cdot\tilde{\eta}_3(-z_{m_{14}})} + \frac{A(2^+,\hat{3}^-,\hat{m}_{23}^r)A(\hat{m}_{14}^k,\hat{4}^\mu,1^+)\eta_3^\mu}{2p_{14}\cdot\tilde{\eta}_3(-z_{m_{14}})}, \quad (26)$$

where

$$-p_4\cdot A^1 \;=\; \frac{1}{\sqrt{2}}\frac{\tilde{\eta}_3\cdot p_4 A(1^+,2^+,\mu)\epsilon_{3-}^\mu}{p_{12}^2} + \frac{1}{\sqrt{2}}\frac{\tilde{\eta}_3^\mu A^1(m_{14}^\mu,\hat{4}^\nu,1^+)p_4^\nu\epsilon_2^+\cdot\epsilon_3^-}{2p_{14}\cdot\tilde{\eta}_3} + \frac{1}{\sqrt{2}}\frac{\tilde{\eta}_3\cdot p_4 A^1(2^+,\hat{3}^+,m_{23}^\mu)\epsilon_{1+}^\mu}{2p_{14}\cdot\tilde{\eta}_3} - i\frac{A^1(2^+,\hat{3}^-,m_{23}^\mu)A^1(m_{14}^\mu,\hat{4}^\nu,1^+)p_4^\nu}{2p_{14}\cdot\tilde{\eta}_3}. \quad (27)$$

C. currents with four on-shell lines

Now we apply our techniques to the currents with four on-shell external line $\mathcal{A}(1^+,2^+,3^+,4^-,\mu)$. To get the full components of the currents, we choose three shifted pairs of lines $(1,5),(2,5),(4,5)$.

For the shifted pair $(1,5)$, the finite poles contributions to the currents come from the Feynman diagrams in Fig. 7. The infinite contributions come from the Feynman diagrams in Fig.8. Then we have

$$\eta_1\cdot A \;=\; -p_5\cdot A^1 + \frac{A(\hat{1}^+,2^+,\hat{m}_{12}^-)A(\hat{m}_{345}^+,3^+,4^-,\hat{5}^\mu)\eta_1^\mu}{2p_{345}\cdot\eta_1(-z_{m_{12}})}, \quad (28)$$

where

$$-p_5\cdot A^1 = \frac{1}{\sqrt{2}}\frac{\eta_1^\mu A(m_{234}^\mu,2^+,3^+,4^-)\epsilon^{\hat{1}^+}\cdot p_5}{p_{234}^2}. \quad (29)$$

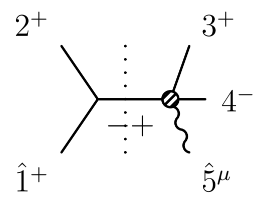

FIG. 7: Finite pole terms in the momentum shift of $(1,5)$

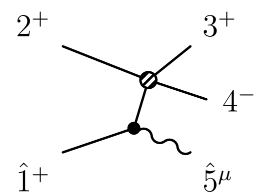

FIG. 8: Boundary terms in the momentum shift of $(1,5)$. Here we use the $\cdot$ to denote the new vertex which is the result of acting $\frac{d}{dz}$ on the original three point vertex.

For the momenta shift of the pair (2,5), we have

$$\begin{aligned}\eta_2 \cdot A \;=\; & -p_5 \cdot A^1 - i\frac{A(1^+,\hat{2}^+,\hat{m}_{12}^-)A(\hat{m}_{345}^+,3^+,4^-,\hat{5}^\mu)\eta_2^\mu}{2p_{345}\cdot\eta_2(-z_{m_{12}})} \\ & - i\frac{A(\hat{2}^+,3^+,\hat{m}_{23}^-)A(\hat{m}_{451}^+,4^-,\hat{5}^\mu,1^+)\eta_2^\mu}{2p_{451}\cdot\eta_2(-z_{m_{23}})} \\ & - i\frac{A(\hat{2}^+,3^+,4^-,\hat{m}_{234}^-)A(\hat{m}_{51}^+,\hat{5}^\mu,1^+)\eta_2^\mu}{2p_{51}\cdot\eta_2(-z_{m_{234}})} \\ & - i\frac{A(\hat{2}^+,3^+,4^-,v)A(\hat{k},\hat{5}^\mu,1^+)\eta_2^\mu}{2p_{51}\cdot\eta_2(-z_{m_{234}})\hat{k}\cdot v},\end{aligned} \tag{30}$$

where

$$\begin{aligned}-p_5\cdot A^1 \;=\; & \sqrt{2}\frac{\eta_2\cdot\epsilon_1^+A^1(m_{234}^\mu,\hat{2}^+,3^+,4^-)p_5^\mu}{2p_{15}\cdot\eta_2} - \frac{1}{\sqrt{2}}\frac{\eta_2\cdot p_5 A^1(m_{234}^\mu,\hat{2}^+,3^+,4^-)\epsilon_1^+}{2p_{15}\cdot\eta_2} \\ & - \frac{1}{\sqrt{2}}\frac{\eta_2^\mu A(m_{34}^\mu,3^+,4^-)\epsilon_{\hat{2}^+}^{\mu_2}A^1(m_{15}^{\mu_2},\hat{5}^{\mu_1},1)p^{\mu_1}}{2p_{15}\cdot\eta_2 p_{34}^2} \\ & - i\frac{A^1(m_{234}^\mu,\hat{2}^+,3^+,4^-)A^1(m_{51}^\mu,\hat{5}^\nu,1^+)p^\nu}{2p_{15}\cdot\eta_2}.\end{aligned} \tag{31}$$

The diagrams contributing to the finite pole terms and boundary terms are shown in Fig.9 and Fig.10 respectively. For the momentum shift of the pair (4,5), we have

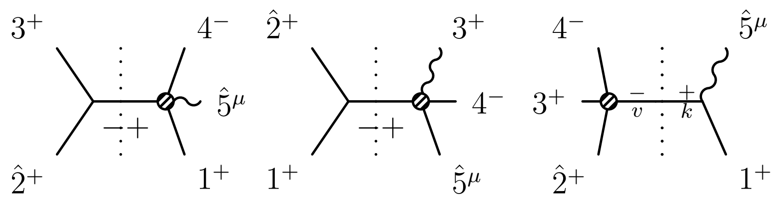

FIG. 9: Finite pole terms in the momentum shift of $(2,5)$

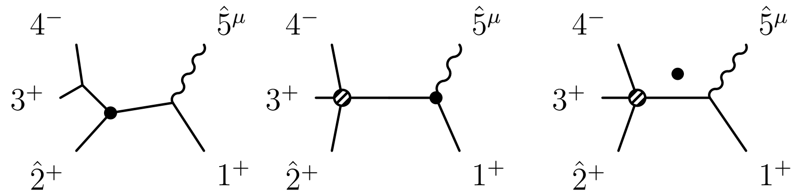

FIG. 10: Boundary terms in the momentum shift of $(2,5)$. Here we use the $\cdot$ to denote that we should act the $\frac{d}{dz}$ on the corresponding three point vertex or propagator first.

$$
\begin{aligned}
\tilde{\eta}_3 \cdot A \;=\; & -p_5 \cdot A^1 - i\frac{\tilde{\eta}_3^\mu A(\hat{5}^\mu, 1^+, \hat{m}_{15}^{\overset{+}{k}}) A(\hat{m}_{234}^{\overset{-}{v}}, 2^+, 3^+, \hat{4}^-)}{2p_{15}\cdot \eta_2(-z_{m_{15}})} \\
& - i\frac{\tilde{\eta}_3^\mu A(\hat{5}^\mu, 1^+, 2^+, \hat{m}_{125}^{\overset{+}{\underset{-}{k}}}) A(\hat{m}_{34}^{\overset{-}{\underset{+}{v}}}, 3^+, \hat{4}^-)}{2p_{125}\cdot \eta_2(-z_{m_{125}})}
\end{aligned}
\tag{32}
$$

and the boundary term is

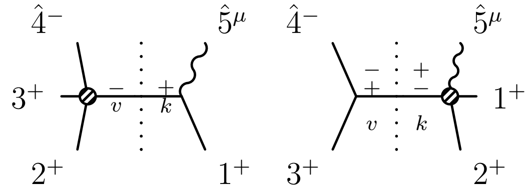

FIG. 11: Finite pole terms in the momentum shift of $(4,5)$

$$
\begin{aligned}
-p_5\cdot A^1 &= \sqrt{2}\frac{\tilde\eta_3\cdot\epsilon_1^+A^1(m_{234}^\mu,2^+,3^+,\hat4^-)p_5^\mu}{2p_{15}\cdot\tilde\eta_3}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3\cdot p_5A^1(m_{234}^\mu,\hat2^+,3^+,\hat4^-)\epsilon_1^+}{2p_{15}\cdot\tilde\eta_3}\\
&+\sqrt{2}\frac{\tilde\eta_3^\mu\cdot A(m_{12}^\mu,1^+,2^+)A^1(m_{34}^\nu,3^+,\hat4^-)p_5^\nu}{2p_{125}\cdot\tilde\eta_3\ p_{12}^2}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3\cdot p_5A^1(m_{34}^\mu,3^+,\hat4^-)A(m_{12}^\mu,1^+,2^+)}{2p_{125}\cdot\tilde\eta_3\ p_{12}^2}\\
&+\sqrt{2}\frac{\tilde\eta_3^\mu\cdot A(m_{123}^\mu,1^+,2^+,3^+)\epsilon_{\hat4^-}\cdot p_5}{p_{123}^2}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3\cdot p_5\epsilon_{\hat4^-}^\mu A(m_{123}^\mu,1^+,2^+,3^+)}{p_{123}^2}\\
&+\sqrt{2}\frac{\tilde\eta_3^\mu\cdot\epsilon_{2^+}A^1(3^+,\hat4^-,m_{34}^{\mu_1})A^1(m_{15}^{\mu_1},\hat5^{\mu_2},1^+)p_5^{\mu_2}}{2p_{125}\cdot\tilde\eta_3\ 2p_{15}\cdot\tilde\eta_3}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3^{\mu_1}A^1(m_{15}^{\mu_1},\hat5^{\mu_2},1^+)p_5^{\mu_2}\epsilon_{2^+}^\mu A^1(3^+,\hat4^-,m_{34}^\mu)}{2p_{125}\cdot\tilde\eta_3\ 2p_{15}\cdot\tilde\eta_3}\\
&+\sqrt{2}\frac{\tilde\eta_3^\mu A(2^+,3^+,m_{23}^\mu)\epsilon_4^{\mu_1}A^1(m_{15}^{\mu_1},\hat5^{\mu_2},1^+)p_5^{\mu_2}}{2p_{23}^2\ 2p_{15}\cdot\tilde\eta_3}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3^{\mu_1}A^1(m_{15}^{\mu_1},\hat5^{\mu_2},1^+)p_5^{\mu_2}\ A(2^+,3^+,m_{23}^\mu)\epsilon_4^\mu}{2p_{23}^2\ 2p_{15}\cdot\tilde\eta_3}\\
&+\sqrt{2}\frac{\tilde\eta_3^\mu\cdot\epsilon_{3^+}\epsilon_{\hat4^-}^{\mu_1}A^1(m_{512}^{\mu_1},\hat5^{\mu_2},1^+,2^+)p_5^{\mu_2}}{2p_{125}\cdot\tilde\eta_3}-\frac{1}{\sqrt{2}}\frac{\tilde\eta_3^{\mu_1}A^1(m_{512}^{\mu_1},\hat5^{\mu_2},1^+,2^+)p_5^{\mu_2}\epsilon_{3^+}\cdot\epsilon_{4^-}}{2p_{125}\cdot\tilde\eta_3}\\
&-i\frac{A^1(m_{234}^\mu,2^+,3^+,\hat4^-)A^1(m_{51}^\mu,\hat5^\nu,1^+)p_5^\nu}{2p_{15}\cdot\tilde\eta_3}\\
&-i\frac{A^1(m_{34}^\mu,3^+,\hat4^-)A^1(m_{512}^\mu,\hat5^\nu,1^+,2^+)p_5^\nu}{2p_{15}\cdot\tilde\eta_3}. \qquad (33)
\end{aligned}
$$

The corresponding diagrams of the finite pole terms and boundary terms are shown in Fig.11, Fig.12, Fig.13 respectively.

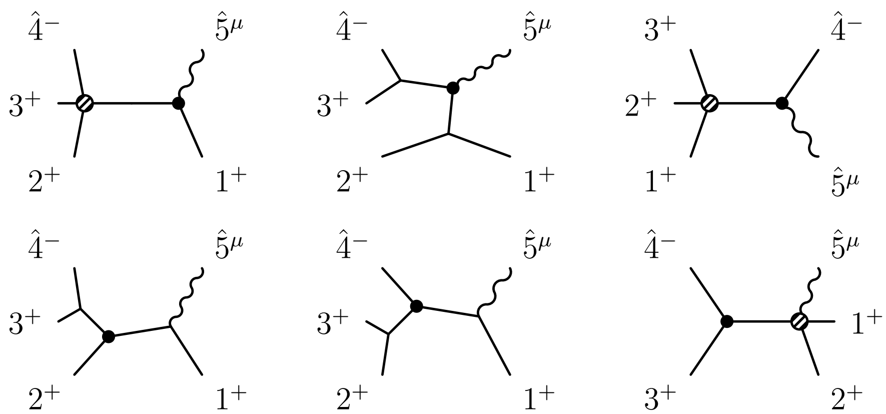

FIG. 12: Boundary terms in the momentum shift of $(4,5)$. Here we use the $\cdot$ to denote that we should act the $\frac{d}{dz}$ on the corresponding three point vertex first.

V. CONCLUSION

In this article, we analysis the cancelation in detail among the terms in Ward identity at tree-level in Feynman gauge. According to this, we prove the complexified Ward

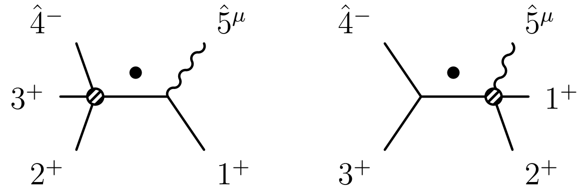

FIG. 13: Boundary terms in the momentum shift of $(4,5)$. Here we use the $\cdot$ to denote that we should act the $\frac{d}{dz}$ on the corresponding propagator first.

identity for the tree-level amplitudes directly. Using the complexified Ward identity, we get exactly the usual BCFW recursion relation when reduced to single poles. Furthermore, we propose a new form for the boundary term for the BCFW shifted amplitude or off-shell vector currents. In this way, the boundary terms can be obtained by a new recursion relation. We apply our technique to the off-shell currents with on-shell lines with different helicity structures. It is easy to see that our technique is more efficient for the currents of general helicity structures for the on-shell lines, complimenting the existent off-shell recursion relation. First of all, the number of the effective diagrams is small. For finite poles contribution, only the propagator along the complex line contributes; while for the boundary terms both the propagator and half parts in the three-point vertex contribute to the vector currents. Four-point gluon interaction needs no consideration. In proceeding the recursion relation, there will be new off-shell currents with some of the on-shell states replaced by their momenta. Such new objects can also be obtained in our techniques.

Although we focus on the one-line off-shell vector currents in gauge theory, the technique from complex Ward identity can be generalize to theories with gauge symmetry spontaneously broken as well as to tensor currents with several off-shell lines. The current with two off-shell line is important for constructing one-loop amplitudes. Another extension is to study the amplitude at one loop level according to the loop level Ward identity. However, the complex Ward identity does not present itself at the loop level, it warrants further study.

Acknowledgement

We thank Edna Cheung, Jens Fjelstad, Konstantin G. Savvidy and Yun Zhang for helpful discussions. This work is funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), NSFC grant No. 10775067, Research

Links Programme of Swedish Research Council under contract No. 348-2008-6049, the Chinese Central Government's 985 Project grants for Nanjing University, the China Science Postdoc grant no. 020400383. the postdoc grants of Nanjing University 0201003020

---

[1] G. Chen, *Ward Identity implied recursion relations in Yang-Mills theory*, [arXiv: 1203.6281].

[2] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering,* Phys. Rev. Lett. **56**, 2459 (1986).

[3] Z. Xu, D. H. Zhang and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, Nucl. Phys. B **291**, 392 (1987).

[4] F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with n Gluons,* Nucl. Phys. B **306**, 759 (1988).

[5] D. A. Kosower, Nucl. Phys. B**335**:23 (1990).

[6] L. J. Dixon, *Calculating scattering amplitudes efficiently*, Bouler TASI **95**, 539-584, [arXiv: hep-ph/9601359].

[7] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252**, 189 (2004), [arXiv: hep-th/0312171].

[8] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, Phys. Rev. D **71**, 025012 (2005) [arXiv:hep-th/0410179].

[9] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, Nucl. Phys. B **725**, 275-305 (2005), [arXiv:hep-th/0412103].

[10] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, Nucl. Phys. B **715**, 499-522 (2005), [arXiv:hep-th/0412308].

[11] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, Phys. Rev. Lett. **94**, 181602 (2005), [arXiv:hep-th/0501052].

[12] S. D. Badger, E. W. Glover, V. V. Khoze and P. Svrcek, *Recursion relations for gauge theory amplitudes with massive particles*, JHEP **0507**, 025 (2005), [arXiv: hep-th/0504159].

[13] K. J. Ozeren and W. J. Stirling, *Scattering amplitudes with massive fermions using BCFW recursion*, Eur. Phys. J. C **48**, 159-168 (2006), [arXiv: hep-ph/0603071].

[14] C. Schwinn, *Twistor-inspired construction of massive quark amplitudes*, Phys. Rev. D **78**, 085030 (2008), [arXiv: 0809.1442].

[15] G. Chen, K. G. Savvidy, *Spinor formalism for massive fields with half-integral spin*, Eur. Phys. J. C **72**, 3, 1952 (2012), [arXiv:1105.3851].

[16] G. Chen, *Recursion relations for the general tree-level amplitudes in QCD with massive*

dirac fields, Phys.Rev. D83 (2011) 125005, [arXiv: 1103.2518].

[17] Z. Bern, L. J. Dixon, David C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, Nucl. Phys. B **425**, 217 (1994) , [arXiv: hep-ph/9403226].

[18] Z. Bern, L. J. Dixon, David C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. B **435**, 59 (1995), [arXiv: hep-th/9409265].

[19] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in N=4 super-Yang-Mills thoery*, Phys. Rev. D **71**, 045006 (2005), [arXiv: hep-th/0410224].

[20] Z. Bern, N. E. J. Bjerrum-Bohr, David C. Dunbar and Harald Ita, *Recursive calculation of one loop QCD integral coefficients*, JHEP 0511 (2005) 027, [arXiv: hep-th/0507019].

[21] Carola F. Berger, Zvi Bern, Lance J. Dixon, Darren Forde, David A. Kosower, *Bootstrapping One-Loop QCD Amplitudes with General Helicities.*, Phys.Rev. D74 (2006) 036009 , [arXiv: hep-ph/0604195].

[22] Nima Arkani-Hamed, Jacob L. Bourjaily, Freddy Cachazo, Simon Caron-Huot, Jaroslav Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM*, JHEP 1101 (2011) 041 , [arXiv: 1008.2958].

[23] B. Feng, Z.B. Zhang, *Boundary Contributions Using Fermion Pair Deformation*, JHEP 1112 (2011) 057, [arXiv: 1109.1887].